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A HYBRID-PERTURBATION-GALERKIN TECHNIQUE

WHICH COMBINES MULTIPLE EXPANSIONS

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Abstract

A two-step hybrid perturbation-Galerkin method for the solution of a variety of differential equations type problems is found to give better results when multiple perturbation expansions are employed. The method assumes that there is a parameter in the problem formulation and that a perturbation method can be used to construct one or more expansions in this parameter. An approximate solution is constructed in the form of a sum of perturbation coefficient functions multiplied by computed amplitudes. In step one, regular and/or singular perturbation methods are used to determine the perturbation coefficient functions. The results of step one are in the form of one *or more* expansions each expressed as a sum of perturbation coefficient functions multiplied by *a priori* known gauge functions. In step two the classical Bubnov-Galerkin method uses the perturbation coefficient functions computed in step one to determine a set of amplitudes which replace and improve upon the gauge functions. The hybrid method has the potential of overcoming some of the drawbacks of the perturbation and Galerkin methods as applied separately, while combining some of their better features. In this study the proposed method is applied, with two perturbation expansions in each case, to a variety of model ordinary differential equations problems including: a family of linear two-point boundary-value problems, a nonlinear two-point boundary-value problem, a quantum mechanical eigenvalue problem and a nonlinear free oscillation problem. The results obtained from the hybrid method are compared with approximate solutions obtained by other methods, and the applicability of the hybrid method to broader problem areas is discussed.

1. INTRODUCTION

A two-step hybrid analysis technique, which combines perturbation techniques with the Galerkin method, was presented and discussed by Geer and Andersen [7], where it was applied to some singular perturbation problems in slender body theory. That technique promises to be useful in the analysis of a very wide variety of differential equations type problems. In this paper we shall further develop the method through the use of multiple perturbation expansions and demonstrate its improved usefulness for several classes of two-point boundary-value problems for ordinary differential equations.

The Galerkin method [6] has, of course, been known and used for a long time. But a principal problem associated with its successful application lies in the choice of appropriate basis functions. This problem was addressed when the hybrid technique was apparently first introduced by Ahmed K. Noor and collaborators in conjunction with the finite element analysis of geometrically nonlinear problems in structural mechanics. In a series of papers [11-21 among others] they have shown for a variety of structural mechanics and thermal problems that the first few terms in a Taylor series expansion of the solution of a parameterized system of discretized equations can be particularly effective as Galerkin trial functions (or basis vectors). Subsequently, Geer and Andersen [7] have shown for two classes of problems in slender body theory that the terms in a *singular* perturbation expansion of the solution are also effective trial functions. It has been demonstrated that the "reduced-basis" solutions can be useful for significantly larger values of the expansion parameter than the Taylor series or singular perturbation solutions on which they are based. A treatment of the reduced basis method from a mathematical point of view is given by Fink and Rheinboldt [5].

Some general observations about the technique are the following. First, in many perturbation problems, much effort has to be expended to compute (analytically) each additional term in a perturbation expansion. Through the use of the proposed hybrid method, the known perturbation terms can be exploited more fully. Secondly, another way of viewing the technique is to recognize that in many perturbation expansions the functional form of the higher order terms can be well approximated by a linear combination of a few lower order terms. Thus, much of the effect of the higher-order terms may be included by using only a small number of lower-order perturbation functions, but with their amplitudes replaced by new functions of the expansion parameter. Thirdly, since it is often possible to develop formal perturbation expansions for the solution about two or more values of the parameter, e. g. for small or large values of the parameter, the proposed method appears to be a convenient way of combining the information contained in these different expansions. The set of basis functions is chosen so as to include terms from the different perturba-

tion expansions. In fact, when information from two or more expansions is employed, the method appears to provide meaningful (and often very accurate) approximations not only in the regions near the expansion points but also in the "intermediate" regions for the parameter, e. g. in regions where the parameter is neither "small" nor "large".

It is our belief that the conjunction of perturbation and Galerkin techniques can be useful in a wide variety of application areas and in these applications the hybrid technique will give better approximations than the perturbation method alone. In this paper we will present applications of the technique to several classes of problems with second order ordinary differential equations. Two perturbation expansions are used in each of the problems illustrated. In the next section the method is described in more detail. In section 3 it is applied to a general class of linear two-point boundary-value problems and in section 4 to a related linear problem. In sections 5, 6, and 7 the method is illustrated by application to a representative nonlinear problem, to a quantum mechanical eigenvalue problem, and to a nonlinear oscillation problem, respectively. In section 8 we make some additional observations about the method, discuss some of the insights it provides, and indicate some areas for further study.

2. DESCRIPTION OF THE METHOD

The method we wish to describe is a two-step hybrid analysis technique. It is based upon the successive use of perturbation expansions and the classical Bubnov-Galerkin approximation technique. In the perturbation methods, an approximation to the solution of a particular problem involving a parameter is developed in terms of a series of unknown functions with preassigned coefficients, i. e. gauge functions. The unknown functions are usually determined by solving a recursive set of differential equations which are, in general, simpler than the original governing differential equation. By contrast, in the Bubnov-Galerkin technique one seeks an approximate solution to the problem in the form of a linear combination of specified (known) coordinate (also called perturbation coordinate) functions with unknown coefficients. The coefficients are determined by demanding that the residual formed by substituting the trial solution into the governing differential equation is orthogonal to each of the coordinate functions.

While both of these methods are useful and have been successful in providing approximate solutions to a wide variety of nonlinear (or otherwise difficult) problems, each has certain drawbacks. The perturbation method has at least two major drawbacks. First, as the number of terms in the perturbation expansion increases, the mathematical complexity of the equations which determine the unknown functions increases rapidly. Thus, in most practical applications the computed perturbation series is limited to only a few terms. A

second drawback to the perturbation method is the requirement of restricting the perturbation parameter to values close to the value about which the expansion was constructed (e. g. to small or large values of the parameter) in order to obtain solutions of acceptable accuracy. (These drawbacks of the perturbation method have been recognized and several modifications or extensions have been proposed, see e. g. Van Dyke [22] and Andersen and Geer [1]). The main shortcoming of the Bubnov-Galerkin method is the difficulty, from a practical point of view, of selecting a small number of good coordinate functions. The Bubnov-Galerkin method usually is not appropriate when a large number of coordinate functions are used because of the lack of sparseness in the generated equations.

To illustrate the general ideas of the hybrid (or "reduced basis") method, suppose we are seeking (an approximation to) the solution u to the problem

$$(2.1) \qquad \mathcal{L}(u, \epsilon) = 0,$$

where $\mathcal{L}$ is some differential operator and ϵ is a parameter. (Although we shall restrict our attention to problems involving second-order ordinary differential equations in this paper, we believe that the method has a much wider range of applicability. Hence, we formulate the method in terms of a general differential operator $\mathcal{L}$ and in section 8 discuss some of the other possible areas of application of the method.) Here (2.1) holds in some domain D, and, in addition, u must satisfy certain conditions on the boundary D. Without loss of generality, we can assume that these boundary conditions are homogeneous in u.

Now the application of the hybrid perturbation-Galerkin method can be divided into the following two distinct steps. First, we generate the coordinate functions in a perturbation expansion of u about one or more specific values of the parameter ϵ, say about $\epsilon = \epsilon_p$, $p = 1, 2, \ldots, P$. We then construct a new approximate solution consisting of a sum of some of these perturbation coordinate functions, each multiplied by an unknown amplitude, and determine these amplitudes by using the Bubnov-Galerkin method.

To describe this idea in more detail, suppose that the solution to (2.1) can be expanded about $\epsilon = \epsilon_p$ into a series of the form

$$(2.2) \qquad u = \sum_{j=0}^{n_p-1} u_j^p \, \alpha_j^p(\epsilon) + O(\alpha_{n_p}^p(\epsilon)),$$

where $\{\alpha_j^p(\epsilon)\}$ is an appropriate asymptotic sequence of gauge functions and each u_j^p can be determined completely by a standard perturbation method (e. g. a composite expansion of inner and outer expansions).

A subset of all of the perturbation functions u_j^p are now chosen as coordinate functions

for the Bubnov-Galerkin technique and an approximation $\tilde{u}$ for u is sought in the form

$$(2.3) \qquad \tilde{u} = \sum_{j=1}^{N} u_j \, \delta_j,$$

where the (unknown) parameters $\delta_j = \delta_j(\epsilon)$ represent the amplitudes of the coordinate functions u_j. Here each u_j is one of the perturbation coordinate functions u_k^p. To determine the unknown amplitudes δ_j, we apply the Bubnov-Galerkin technique to the governing equation (2.1). Thus, we substitute (2.3) into (2.1) and demand that the residual be orthogonal to the N coordinate functions u_j over the domain D, i.e.

$$(2.4) \qquad \int_D \mathcal{L}(\sum_{j=1}^{N} u_j \, \delta_j, \epsilon) \, u_k \, dx = 0, \quad k = 1, 2, \ldots, N.$$

Equations (2.4) represent a set of equations for the unknown amplitudes δ_j. While (2.4) must, in general, be solved numerically, doing so is much simpler than numerically solving (2.1). In particular, for a fixed value of ϵ, the solution to (2.4) is a point in N-dimensional space, where N is reasonably small, while the solution of (2.1) is a continuous function.

We should note that this particular choice of coordinate functions overcomes the main drawback of the Bubnov-Galerkin method. By the way they are constructed, the perturbation coordinate functions are (under certain assumptions) elements of a set of functions which span the space of solutions in a neighborhood of their point of generation. Thus, they should fully characterize the solution u in that neighborhood. Also, in many applications, the functions u_j^p are determined by solving a set of *linear* equations, even though the original operator $\mathcal{L}$ may be nonlinear. The first property (completeness) is necessary for the convergence of the Bubnov-Galerkin method, while the second property (linearity) enhances the effectiveness of the proposed hybrid method for solving nonlinear problems.

Another important property of the proposed method is that the coordinate functions, i.e. the perturbation functions, *do not need to come from a regular perturbation expansion.* In fact, all that is needed is a formal asymptotic expansion of the solution to (2.1) for values of ϵ near ϵ_p in the form of (2.2), where $\{\alpha_j^p(\epsilon)\}$ is a sequence of appropriate gauge functions, e. g. expressions which may involve $\log(\epsilon)$ and/or fractional powers of ϵ. Thus, the *proposed method may be applied to singular as well as regular perturbation problems.*

Still another important property of the proposed method is its ability to combine information about the solution from several different asymptotic expansions. As we shall see in the following sections, the hybrid solution, at least in all of the examples we examined, has the nice property of providing reasonable (and often, very accurate) approximations to the solution for all values of the parameter which lie between the points about which the formal asymptotic expansions were constructed.

We shall now illustrate the hybrid method by applying it to several different types of model two-point boundary-value problems with second-order equations.

3. LINEAR EQUATIONS

In this section, we shall apply the hybrid method to a general linear second order problem of the form

$$(3.1) \qquad \epsilon\, u'' + a(x)\, u' + b(x)\, u = c(x), \quad 0 < x < 1,$$

$$u(0) = 0 = u(1).$$

For simplicity, we shall assume that $a(x) > 0$ on $0 \le x \le 1$, so that, for small positive values of ϵ, the boundary layer will form near $x = 0$.

Using standard perturbation techniques (see, e.g. [10]), it is possible to construct formal asymptotic expansions of the solution to problem (3.1) for both small or large values of ϵ. Thus, following the notation of the previous section, we shall consider expansions of u about $P = 2$ values of ϵ, namely, $\epsilon_1 = 0$ and $\epsilon_2 = \infty$. In particular, for small values of ϵ, the expansion is of the form (2.2) with $\alpha_j^1 = \epsilon^j$, and

$$(3.2) \quad u_j^1 = y_j(x) + Y_j(x/\epsilon) - \sum_{i=0}^{j} Y_{j-i}^{(i)}(0)(x/\epsilon)^i/i! - x\left[Y_j(1/\epsilon) - \sum_{i=0}^{j} Y_{j-i}^{(i)}(0)(1/\epsilon)^i/i!\right].$$

The first term on the right side of (3.2) is the jth term in the outer expansion of (3.1), while the second term is the jth term in the inner expansion. The next term is the jth term in the outer expansion of the inner solution. Thus, the first three terms together represent the jth term in a uniform (composite) expansion of the solution for small values of ϵ. This j^{th} term vanishes at $x = 0$, but does not vanish at $x = 1$, where it is exponentially small. Thus, the final term, which is exponentially small when ϵ is small, is added to insure that u_j^1 vanishes identically at both 0 and 1. In particular, we find

$$(3.3) \qquad \begin{aligned} y_o(x) &= \frac{\int_1^x (c(t)/a(t))E(t)\,dt}{E(x)}, \\ E(x) &= e^{\int_1^x (b(t)/a(t))\,dt}, \\ Y_0(x/\epsilon) &= Y_0(0)\left\{1 - e^{-a(0)\,x/\epsilon}\right\}. \end{aligned}$$

Other terms can be computed in a straightforward manner (see, e.g., [10]).

For large values of ϵ, the expansion of the solution of (3.1) is again of the form (2.2) with $\alpha_j^2 = \epsilon^{-j-1}$, and

$$(3.4) \qquad u_j^2 = z_j(x) = F_j(x) - xF_j(1).$$

$$F_j(x) = \int_0^x \int_0^s f_j(t)\, dt\, ds,$$

$$f_0(x) = c(x), \qquad f_j(x) = -a(x)\, z'_{j-1}(x) - b(x)\, z_{j-1}(x), \qquad j \geq 1.$$

We now use these two expansions to construct a hybrid solution to (3.1). In particular, we select n_1 terms from the small-ϵ expansion (3.2) and n_2 terms from the large-ϵ expansion (3.4) and write our hybrid solution in the form (2.3), with $N = n_1 + n_2$. We then substitute this expression into (3.1) and use the orthogonality conditions (2.4) to obtain the conditions

$$(3.5) \qquad \sum_{j=1}^{N} A_{kj}(\epsilon)\, \delta_j(\epsilon) = B_k, \qquad k = 1, 2, \ldots, N$$

$$A_{kj} = \int_0^1 (\epsilon\, u''_j + a\, u'_j + b\, u_j)\, u_k\, dx, \qquad B_k = \int_0^1 c\, u_k\, dx.$$

Thus, we see that the δ_j's satisfy a set of linear algebraic equations, which, of course, follows from the fact the original problem (3.1) is linear. Once these equations have been solved, the values of the δ_j's can be used along with the corresponding (coordinate) functions u_j in (2.3) to yield our hybrid solution.

To illustrate our method for this class of problems, we apply it to the following three model problems. The first problem is

$$(3.6) \qquad \epsilon\, u'' + u' + u = \sin(2\pi x),$$

$$u(0) = 0 = u(1).$$

In Figure 1, we have plotted for $\epsilon = 0.2$ in equation (3.6), (a) the singular perturbation solution $P_0[1]$ based on the one-term expansion about $\epsilon = 0$, (b) the regular perturbation solution $P_\infty[1]$ based on the one-term expansion about $\epsilon = \infty$, (c) the hybrid solution H[1,1] which unifies the two perturbation expansions, and (d) the "exact" solution computed by a numerical shooting method. (Here the notation $H[n_1, n_2]$ stands for a hybrid solution based on n_1 terms from the first perturbation expansion and n_2 terms from the second expansion.) While neither one-term perturbation expansion succeeds well near $\epsilon = 0.2$, the hybrid solution based on just two perturbation coefficient functions gives a reasonable approximation not only at $\epsilon = 0.2$ but, in fact, for all positive values of ϵ.

The second problem is

$$(3.7) \qquad \epsilon\, u'' + e^{-x}\, u' + e^x\, u = 1,$$

$$u(0) = 0 = u(1).$$

In Figure 2, we have plotted for $\epsilon = 0.2$ in equation (3.7) comparable curves to those shown in Figure 1. Again while neither one-term perturbation expansion succeeds well

near $\epsilon = 0.2$ the hybrid solution based on just two perturbation coefficient functions again gives a reasonable approximation for all positive values of ϵ.

To illustrate a problem wherein the $H[1,1]$ solution is not adequate we consider the equation

$$(3.8) \qquad \epsilon u'' + e^x u' + u = 1 + e^{-x} \sin(2\pi x),$$

$$u(0) = 0 = u(1).$$

In Figure 3, we have plotted for $\epsilon = 0.3$ in equation (3.8) the following: (a) the singular perturbation solutions $P_0[1]$ and $P_0[2]$ based on the one-term and two-term expansions, respectively, about $\epsilon = 0$, (b) the regular perturbation solutions $P_\infty[1]$ and $P_\infty[2]$ based on one-term and two-term expansions about $\epsilon = \infty$, (c) the hybrid solutions $H[1,1]$ and $H[2,2]$, and (d) the "exact" solution. Here $H[1,1]$ gives a poor approximation for $\epsilon = 0.3$. However the use of one additional perturbation coefficient function from each expansion results in a reasonable approximation at $\epsilon = 0.3$ and in fact for all positive values of ϵ. We shall return to these examples in section 8.

4. A SPECIAL SINGULAR LINEAR PROBLEM

The hybrid method can be applied successfully to a wider range of linear problems than considered in the previous section. To illustrate this point we shall apply the method to the model problem

$$(4.1) \qquad \epsilon u'' + x u' - x u = x(a + bx), \qquad 0 < x < 1,$$

$$u(0) = 0 = u(1),$$

where a and b are specified constants. (See Hanks [8] where this equation is used to model heat flow near an ocean rise.) In (4.1) the coefficient of u' vanishes at $x = 0$. Thus, even though a boundary layer still forms near $x = 0$, the formal perturbation solution for small positive values of ϵ contains fractional powers of ϵ as well as $\log(\epsilon)$.

For small values of ϵ it is straightforward to show that

$$u(x, \epsilon) = u_0^1(x, \epsilon) + O(\epsilon^{1/2}) + O(\epsilon \log(\epsilon)) \quad \text{as} \quad \epsilon \to 0,$$

$$u_0^1(x, \epsilon) = \hat{u}_0(x, \epsilon) - x\hat{u}_0(1, \epsilon),$$

$$(4.2)$$

$$\hat{u}_0(x, \epsilon) = -bx + \left(\tfrac{a+2b}{\epsilon}\right)(e^x - 1) + C \int_0^{x/\sqrt{\epsilon}} e^{-t^2/2}\, dt$$

$$C = \sqrt{2/\pi} \left\{ \tfrac{a+2b}{\epsilon} - a - b \right\}.$$

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In (4.2) $\hat{u}_0$ is the first term in a composite expansion based upon a one-term inner expansion and a one-term outer expansion. Then u_0^1 is obtained by subtracting from $\hat{u}_0$ the (exponentially small) term $x\,\hat{u}_0(1,\epsilon)$ to ensure that the boundary conditions are satisfied exactly.

For large values of ϵ, u has a perturbation expansion in the form of (2.2) with $\alpha_j^2 = \epsilon^{-j-1}$. The coefficients u_j^2 are again given by (3.4) with $a(x) = x$, $b(x) = -x$, and $c(x) = x\,(a+b\,x)$.

We can now apply our hybrid method in the manner outlined in the previous section and obtain (linear) equations of the form (3.5) to determine the coefficients δ_j. In Figure 4, we have plotted (for $\epsilon = 0.3$) the approximate solutions to (4.1) based upon a one-term small-ϵ perturbation solution (from equations (4.2)), a one-term large-ϵ perturbation solution (from equations (3.4)), the hybrid solution H[1,1] based on these two one-term expansions, and a solution obtained by a numerical shooting method. Clearly, the hybrid solution is more accurate than either perturbation solution alone. In fact, the value of ϵ used in this figure is very near the value of ϵ corresponding to the poorest agreement between the hybrid and numerical solutions (in the relative L_2 sense) for all positive values of ϵ. We shall discuss this example further in section 8.

5. A NONLINEAR EXAMPLE

To illustrate the application of the hybrid method to a nonlinear problem, we consider the problem of determining the excess pressure $u(x)$ under a slider bearing, which was discussed for large bearing numbers by DiPrima [4]. $u(x)$ satisfies

$$(5.1) \qquad \epsilon \frac{d}{dx}\left\{ h^3(x)[1+u(x)]\frac{du}{dx} \right\} = \frac{d}{dx}\{h(x)[1+u(x)]\}, \quad 0 < x < 1,$$

$$u(0) = 0 = u(1).$$

In (5.1) ϵ is the inverse of the bearing number and $h(x)$ is the (prescribed) film thickness under the bearing, which has been normalized so that $h(1) = 1$. For small values of ϵ DiPrima constructed an asymptotic expansion of the form (2.2) with $\epsilon_1 = 0$ and $\alpha_j^1 = \epsilon^j$, $j = 0, 1, \ldots$. In particular, using his results, we find

$$(5.2) \qquad u_0^1 = \frac{h(0)}{h(x)} + Q\left(\frac{1-x}{\epsilon}\right) - h(0) - (1-x)\left[Q\left(\frac{1}{\epsilon}\right) - h(0) + 1\right],$$

$$h(0)\log\left|\frac{h(0) - 1 - Q}{h(0) - 1}\right| + Q = -\frac{1-x}{\epsilon}.$$

The first three terms in the expression for u_0^1 represent the first term in a (uniform) composite expansion of the solution, while the last term, which is exponentially small for

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small values of ϵ is added to ensure that u_0^1 satisfies the boundary conditions exactly. The second expression in (5.2) defines Q implicitly as a function of $(1-x)/\epsilon$.

For large values of ϵ, u has a regular perturbation expansion of the form (2.2) with $\alpha_j^2 = \epsilon^{-j-1}$. In particular, we find

(5.3)
$$u_0^2 = \int_0^x h^{-2}(t)dt + C\int_0^x h^{-3}(t)dt,$$

$$C = -\int_0^1 h^{-2}(t)dt \Big/ \int_0^1 h^{-3}(t)dt.$$

More terms in this expansion can be computed in a straightforward manner using simple quadratures.

We now select n_1 terms from the small-ϵ expansion of u and n_2 terms from the large-ϵ expansion and form our hybrid solution $\tilde{u}$ in the form (2.3), where $N = n_1 + n_2$. The equations (2.4) for the amplitudes $\{\delta_j\}$ are now quadratically nonlinear. However, we note that, for small values of ϵ, the δ_j's corresponding to small-ϵ perturbation coordinate functions should be approximately ϵ^j, while the δ_j's corresponding to the large-ϵ perturbation coordinate functions should be approximately zero. Thus, these equations can be solved easily (and efficiently) by using Newton's method, starting at small values of ϵ and then proceeding to larger values of ϵ.

To illustrate the accuracy of our hybrid method, we apply it first to the case of a linear film thickness (i.e., a wedge slider) described by $h(x) = 1 + a(1-x)$. In Figure 5, we have plotted the pressure distribution obtained from (5.1) by using a one-term small-ϵ perturbation expansion $P_0[1]$, a one-term large-ϵ expansion $P_\infty[1]$, the hybrid solution $H[1,1]$ based on the two coordinate functions contained in these one-term expansions, and a solution obtained by a numerical shooting method. As can be seen, the agreement between the hybrid solution and the numerical solution is excellent, and the hybrid solution is considerably better than either the small-ϵ or large-ϵ perturbation solutions alone. In fact, as we performed similar comparisons with this example for other values of ϵ, we found that the value of $\epsilon = 0.7$, corresponding to Figure 5, represented the "poorest" agreement between the hybrid and numerical solutions, with a relative L_2-error of about 1%. In Figure 6, we have plotted the hybrid solution $H[1,1]$ and the numerical solution for several different values of ϵ. For values of ϵ substantially larger or smaller than those shown in the figure, the relative L_2-error is considerably less than 1%. In Figure 7, we have plotted δ_1 and δ_2 (the coefficients of the small- and large-ϵ coordinate functions, respectively) as functions of ϵ. As the figure illustrates, δ_1 approaches 1 and δ_2 approaches 0 as ϵ approaches zero, while δ_2 dominates δ_1 as ϵ approaches infinity.

In Figure 8, we have made a similar comparison to that shown in Figure 5, except that the film thickness corresponds to a parabolic shape as discussed by DiPrima. The

results depicted in Figure 8 are typical of those we obtained for several different parabolic shapes. For the shape chosen, the value of ϵ used in the figure corresponds to the "poorest" agreement between the hybrid and numerical solutions.

In section 8, we shall discuss some of the insights this example provides for the application of the hybrid method to other nonlinear problems.

6. AN EIGENVALUE PROBLEM

We feel that our hybrid method can be particularly useful when applied to eigenvalue problems and we are currently investigating its application to some general classes of such problems. However, for our present purposes, we shall be content with illustrating its application to second order eigenvalue problems by considering the following (quantum mechanical type) model problem:

$$\mathcal{L}(u, \epsilon, \lambda) = u''(x) + \{\lambda - V(x, \epsilon)\} u(x) = 0, \qquad -\infty < x < \infty,$$

(6.1)
$$u \to 0 \text{ as } x \to \pm\infty, \qquad \int_{-\infty}^{\infty} u^2(x)\, dx = 1,$$

$$V(x, \epsilon) = x^2 + \epsilon\, x^4.$$

Here λ is an eigenvalue and V is a potential function. When $\epsilon = 0$, (6.1) reduces to a well-known quantum mechanical harmonic oscillator problem with a quadratic potential $V(x, 0) = x^2$. For positive values of ϵ we expect to find eigenvalues and corresponding eigenfunctions which reduce to those of the harmonic oscillator as ϵ goes to zero. However, for negative values of ϵ, there are no solutions to the problem (6.1), because V becomes negative for large values of x. Consequently, we expect that the classical perturbation solution to (6.1) will have a zero radius of convergence. This is indeed the case, as we shall now illustrate.

For small values of ϵ, the classical perturbation solution to (6.1) is of the form (2.2), i.e.,

(6.2)
$$u = \sum_{j=0}^{N-1} u_j(x)\, \epsilon^j + O(\epsilon^N), \quad \lambda = \sum_{j=0}^{N-1} \lambda_j\, \epsilon^j + O(\epsilon^N).$$

Substituting (6.2) into (6.1), we find

(6.3)
$$\lambda_0 = 2k + 1, \quad u_0(x) = e^{-x^2/2}\, h_k(x), \quad k = 0, 1, 2, \ldots,$$

$$\lambda_j = \int_{-\infty}^{\infty} f_j(x)\, u_0(x)\, dx, \quad u_j(x) = p_j(x)\, e^{-x^2/2},$$

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$$f_j(x) = x^4 u_{j-1}(x) - \sum_{i=1}^{j-1} \lambda_i u_{j-i}(x), \quad j \geq 1.$$

In (6.3), $h_k(x)$ is the Hermite polynomial of order k and p_j is a polynomial of order $4j + k$. For our purposes here, we shall only consider perturbations about the ground state eigenvalue $\lambda = 1$ corresponding to $k = 0$. Similar results hold for the higher eigenvalues and some of these results are reported and discussed in [2]. In particular, using the symbolic manipulation system MACSYMA [9], we find (for $k = 0$)

(6.4) $$\lambda = 1 + (3/4)\epsilon - (21/16)\epsilon^2 + (333/64)\epsilon^3 - (30885/1024)\epsilon^4 - \cdots,$$

$$h_0(x) = 1/\pi^{1/4}, \quad p_1(x) = -(4x^4 + 12x^2 - 9)/(32\pi^{1/4}),$$

$$p_2(x) = (48x^8 + 416x^6 + 1272x^4 + 3384x^2 - 4677)/(6144\pi^{1/4}).$$

(Several more terms in this series have been computed using a simple BASIC program. Both the ratio and root tests clearly indicate a zero radius of convergence for the perturbation expansion.)

We now apply our hybrid method to this problem using at first only terms from the small-ϵ perturbation solution (6.2). Thus we look for an approximate solution for u of the form (2.3), where, for convenience, we now begin the summation at $j = 0$ and let the upper limit be $N - 1$. Then each u_j is defined in (6.3) with $k = 0$. Substituting this expression into (6.1), the orthogonality conditions (2.4) can be written as the matrix-vector eigenvalue problem

(6.5) $$(C + \lambda D)\,\vec{\delta} = 0,$$

where $\vec{\delta}$ is the vector of the unknown amplitudes δ_j and the N by N matrices C and D are defined by

(6.6)
$$C_{ij} = \int_{-\infty}^{\infty} \left\{ u_j''(x) - (x^2 + \epsilon\, x^4)\, u_j(x) \right\} u_i(x)\, dx,$$
$$D_{ij} = \int_{-\infty}^{\infty} u_i(x)\, u_j(x)\, dx, \qquad i, j = 0, 1, \ldots, N - 1.$$

Thus, for a fixed value of ϵ, the smallest eigenvalue of (6.5) will provide our approximation to the smallest eigenvalue of (6.1), while the corresponding eigenvector, when used in (2.3), will provide our approximation to the corresponding eigenfunction.

In Figure 9, we have plotted the approximations to the ground state eigenvalue for different values of ϵ obtained by using $N = 1, 2, \ldots, 6$ terms in the perturbation and hybrid solutions, along with the eigenvalues obtained by a purely numerical solution to (6.1). The

figure indicates that the perturbation solution is essentially useless for approximating the eigenvalues, except for extremely small values of ϵ. It also indicates that for any fixed value of ϵ the hybrid approximations appear to be converging to the correct eigenvalue as N increases, although, as expected, the convergence is slower for larger values of ϵ. Incidentally, the two-term perturbation solution is identical to the hybrid solution with $N = 1$.

To find approximations to the solution of (6.1) valid for large values of ϵ, it is convenient to introduce the new variables $\mu = \epsilon^{-1/3}\lambda$, $z = \epsilon^{1/6}x$, and $v(z) = \epsilon^{1/12}u(x)$. Then (6.1) becomes

$$(6.7) \qquad v''(z) + \left\{\mu - z^4 - \epsilon^{-2/3}z^2\right\}v(z) = 0,$$

$$v \to 0 \quad \text{as} \quad z \to \pm\infty, \qquad \int_{-\infty}^{\infty}v^2(z)dz = 1.$$

Then $v(z,\epsilon)$ has a perturbation solution in the form of (2.2) with $\alpha_j^2(\epsilon) = \epsilon^{-2j/3}$, with a similar expansion holding for $\mu = \mu(\epsilon)$. In particular, the leading term in this expansion satisfies the problem

$$v_0''(z) + \left\{\mu_0 - z^4\right\}v_0(z) = 0$$

$$(6.8) \qquad v_0 \to 0 \quad \text{as} \quad z \to \pm\infty, \qquad \int_{-\infty}^{\infty}v_0^2(z)dz = 1.$$

Although we have not been able to express the solution to (6.8) in closed form, it can be solved numerically in a straightforward manner. In particular, we find that $\mu_0 \cong 1.0604$, which implies that $\lambda = 1.0604\,\epsilon^{1/3} + O(\epsilon^{-1/3})$ as $\epsilon \to \infty$.

We now add the function $v_0(z)$ to the set of basis functions formed in (6.3) and construct some new hybrid approximations. That is, we use $v_0(z)$ as well as some of the small-ϵ perturbation coordinate functions in our approximation (2.3). In Figure 10 we have plotted the eigenvalues obtained from the one-term large-ϵ perturbation solution $P_\infty[1]$ as well as those obtained from the hybrid solutions $H[0,1], H[1,0]$, and $H[1,1]$. As the figure clearly indicates, the inclusion of one term from the large-ϵ perturbation expansion dramatically improves the accuracy of our approximations.

The quality of the hybrid approximations to the corresponding ground state eigenfunction is illustrated in Figure 11. Here the hybrid approximations based on different numbers of terms from the small-ϵ and large-ϵ perturbation solutions are plotted along with the eigenfunction obtained by a numerical shooting method for a fixed value of ϵ. Again, the hybrid approximations based on just the small-ϵ perturbation solution, i.e., $H[M,0]$, are apparently converging as M increases, while the inclusion of one term from the large-ϵ expansion again dramatically increases the accuracy of our hybrid approximation.

7. A NATURAL FREQUENCY CALCULATION

The hybrid method we are proposing can also be useful in determining approximations to the period T (or, equivalently, the natural frequency $2\pi/T$) for a nonlinear oscillator involving a parameter ϵ. To illustrate applications to problems in this area, we consider the problem of determining the natural frequency of a simple mechanical system consisting of a mass m restrained by two identical springs, each having natural length L and spring constant k, midway between two parallel walls a distance $2d$ apart, as discussed by Arnold and Case [3]. Thus, if we let h be the maximum displacement of the mass, the displacement along the centerline between the two planes is described by $h\,u(x)$, where

$$(7.1) \qquad \nu^2 u'' + u + \mu u \left\{ 1 - \frac{1}{\sqrt{\epsilon^2 u^2}} \right\} = 0,$$

$$u = 1, \quad u' = 0 \quad \text{at} \quad x = 0; \quad u(x + 2\pi) = u(x).$$

In (7.1), $\epsilon = h/d$, $\mu = \lambda/(1 - \lambda)$, $\lambda = L/d < 1$, $\nu = 2\pi/Tw$, $w = \sqrt{2k(1 - \lambda)/m}$ and $x = w\nu t$, where t is time.

For small values of ϵ, u has an expansion of the form (2.2), with $\alpha_j^1 = \epsilon^{2j}$, and a similar expansion holds for ν^2. In particular, using the Lindstedt-Poincaré method (see, e.g. Nayfeh [10]), we find

$$(7.2) \qquad u = \cos(x) + (\mu/64) \left\{ \cos(3x) - \cos(x) \right\} \epsilon^2 + O(\epsilon^4)$$

$$\nu^2 = 1 + (3\mu/8)\epsilon^2 + O(\epsilon^4) \quad \text{as} \quad \epsilon \to 0.$$

For large values of ϵ, u has a (composite) expansion in the general form of (2.2) with $\alpha_j^2 = \epsilon^{-j}$, in which the formal outer expansion, valid for $\epsilon u > 1$, must be supplemented by inner expansions around $x = \pi/2$ and $3\pi/2$, where u vanishes. In this case, it is easy to show that u is still well approximated by the first term on the right side of (7.2), while

$$(7.3) \qquad \nu^2 = 1 + \mu + O(\epsilon^{-1}) \quad \text{as} \quad \epsilon \to \infty.$$

To apply our hybrid method to this problem, we look for an approximate solution $\tilde{u}$ for u in the form

$$(7.4) \qquad \tilde{u} = \sum_{j=0}^{N} u_j(x)\,\delta_j, \qquad \delta_0 = 1,$$

where $u_0(x) = \cos(x)$ and the remaining $u_j(x)$ can be selected from either the small-ϵ or large-ϵ expansions of u. Substituting (7.4) into (7.1), we see that the initial conditions and the periodicity condition are satisfied, while the orthogonality conditions (2.4) become

$$(7.5) \qquad F_j(\nu^2, \delta_1, \delta_2, \ldots, \delta_N) = 0, \quad j = 0, 1, \ldots, N,$$

$$F_j = \int_0^{2\pi} \left\{ \nu^2\, \tilde{u}'' + \tilde{u} + \mu\, \tilde{u}\left(1 - \frac{1}{\sqrt{1 + \epsilon^2\, \tilde{u}^2}}\right) \right\} u_j\, dx.$$

Equations (7.5) are a system of $N + 1$ equations for the $N + 1$ unknowns ν^2 and δ_j, $j = 1, 2, \ldots, N$, which can be solved using Newton's method. For the special case when $N = 0$, we can solve (7.5) explicity and find

$$(7.6) \qquad \nu^2 = 1 + \frac{\mu}{\pi} \int_0^{2\pi} \left\{ 1 - \frac{1}{\sqrt{1 + \epsilon^2 \cos^2(x)}} \right\} \cos^2(x)\, dx.$$

If we can expand (7.6) for small values of ϵ, we recover the expansion for ν^2 in (7.2), while if we expand it for large values of ϵ we recover the expansion (7.3). In Figure 12, we have plotted ν^2 determined by (7.6) as a function of ϵ, and have also plotted selected values determined by purely numerical means. As the figure indicates, the agreement between the hybrid and numerical results is excellent.

8. OBSERVATIONS AND CONCLUDING REMARKS

Sections 3 through 7 illustrate that the hybrid perturbation-Galerkin method unifying two perturbation expansions can successfully be applied to a wide variety of two-point boundary-value problems.

In Section 3 a whole class of linear problems is treated. Three examples are chosen from this class for purposes of illustration. In each problem a singular perturbation expansion is constructed about $\epsilon = 0$, a regular expansion is constructed for large values of ϵ, and the hybrid method combines terms from both expansions. For many problems in this class, as exemplified by equation (3.6) and Figure 1, or equation (3.7) and Figure 2, the $H[1,1]$ type solutions provide reasonable approximations for all positive values of ϵ. Problems where the $H[1,1]$ approximation performs poorly seem to be somewhat unusual. Equation (3.6) and Figure 3 represent just such a case. For $\epsilon = 0.3$, Figure 13 compares hybrid solutions given by $H[2,0], H[1,1]$, and $H[0,2]$ and an "exact" solution determined numerically. In this case $H[0,2]$ is the best of the three hybrid solutions. This is fortunate, since the regular perturbation functions used in $H[0,2]$ are much easier to compute than the singular perturbation functions needed in $H[2,0]$. However, it is unfortunate in that we did not yet know how to determine *a priori* which of these approximations will have better accuracy.

To compare the accuracy of various perturbation and hybrid solutions the corresponding relative L_2-error norms are shown in Figure 14 as functions of ϵ. The typical behavior is

(i) the hybrid solution $H[N,0]$ for each $N = 0,1,2,\ldots$ is more accurate than the perturbation solution $P_0[N]$ upon which it is based;

(ii) both $H[N,0]$ and $P_0[N]$ tend to be more accurate for small ϵ than for large ϵ;

(iii) the hybrid solution $H[0,N]$ for each N is more accurate than the perturbation solution $P_\infty[N]$ upon which it is based;

(iv) both $H[0,N]$ and $P_\infty[N]$ tend to be more accurate for large ϵ than for small ϵ; and

(v) hybrid solutions based on two expansions, such as $H[N,N]$, tend to be accurate everywhere for sufficiently large N.

The problem (3.6) seems to be somewhat atypical. For instance, it is observed that $P_\infty[1]$ outperforms $H[0,1]$ and even $H[1,1]$ for some values of ϵ. However, $H[0,2]$ is much better than $P_\infty[2]$ and $H[2,2]$ is much better that $H[0,2]$ or $H[2,0]$.

In Section 4 we examined a linear problem whose small-ϵ perturbation expansion involves fractional powers of ϵ as well as terms involving log (ϵ). For this example, the results of the hybrid method were *still* very good, with the hybrid solution based upon just one term from each of the two perturbation expansions providing a good approximation to the exact solution. This example provides a deeper insight into our method in the following way. If we consider a hybrid solution based on only *one* term from the small-ϵ expansion, i.e. $\tilde{u} = \delta_1 u_0^1(x,\epsilon)$, where $u_0^1(x,\epsilon)$ is defined in (4.2), then from (3.5) we see that $\delta_1 = B_1/A_{11}$, were B_1 and A_{11} are defined in (3.5). Using the definition of $u_0^1(x,\epsilon)$ in this expression, we find that δ_1 has an expansion for small ϵ in the form of a series in powers of $\epsilon^{1/2}$. Thus, some of the higher order ϵ dependence of the solution is "anticipated" by δ_1. Moreover, if we consider a hybrid solution based on *two* terms from the small-ϵ expansion, i.e. if we include the coefficient $u_1^1(x,\epsilon)$ of $\epsilon^{1/2}$ in (4.2) in our set of basis functions, we find that both δ_1 and δ_2 have expansions for small ϵ that not only involve powers of $\epsilon^{1/2}$ but also involve $\log(\epsilon)$. This last fact stems from the fact that $u_1^1(x,\epsilon)$ involves an integral whose integrand behaves like x^{-1} near $x = 0$ and whose lower limit of integration is proportional to $\epsilon^{1/2}$. Thus, again, some of the higher order singular behavior of the solution seems to be accounted for in the hybrid coefficients.

Since the class of nonlinear differential equations is too broad for a general treatment, a specific nonlinear differential equation was selected for treatment in Section 5. The hybrid method again combines a singular perturbation expansion about $\epsilon = 0$ with a regular expansion valid for large ϵ. Very good accuracy is obtained by using only one term from each of the two expansions, but, of course, better accuracy is expected if more terms are used. As is generally the case for nonlinear problems, the equations which determine the δ_j are nonlinear and require a numerical solution. However, these equations can be solved efficiently using Newton's method. For small values of ϵ the form of the δ_j is known. Thus,

there is a good starting point for ϵ near zero and one can incrementally increase ϵ, solving for the δ_j one step at a time.

An eigenvalue problem arising in quantum mechanics is discussed in Section 6. For this problem the well known perturbation expansion at $\epsilon = 0$ of the ground state wave function consists of terms which are polynomials in x multiplied by $\exp(-x^2/2)$. However, this expansion has zero radius of convergence, and perturbation expansions of order greater than one are essentially useless. On the other hand, the hybrid solutions based on a few terms in this divergent expansion (see Figure 9) give accurate results for ϵ from 0 up to order of magnitude one. While the hybrid solutions appear to converge for all ϵ, the rate of convergence is slower as ϵ increases, so the technique, while a decided improvement over the perturbation method, is, in fact, useless for high values of ϵ. In Andersen and Geer [2] several low energy states were examined with this method and the same conclusions apply except that the convergence is observed to be slower for higher energy levels. A perturbation expansion for large values of ϵ can also be determined even though none of its terms can be expressed explicitly in terms of elementary functions. However, they can be determined numerically. Hybrid solutions based on just the first term of the expansion at $\epsilon = \infty$ and on one or more terms of the expansion at $\epsilon = 0$ give results which are far more accurate than the hybrid solutions based solely on terms from the expansion about $\epsilon = 0$ (see Figures 9 and 10). The use of the hybrid method with the inclusion of one term from the large-ϵ expansion gives useful approximations for all positive values of ϵ.

Finally, in Section 7 there is a natural frequency calculation for a mechanical system consisting of a mass and two springs. The perturbation expansions are performed for small and large amplitudes of the oscillations. Again by using only one term from the small-amplitude expansion and one term from the large-amplitude expansion, a hybrid solution is constructed which yields better approximations for the natural frequency than do the perturbation solutions on which it is based.

Although we do not as yet have a general proof that our hybrid method will converge to the exact solution as the total number of terms N in the solution increases, we are currently working on such a proof for general classes of linear problems, such as those considered in section 3. As a preliminary step in this direction, a simple linear example with constant coefficients was analyzed in depth in Andersen and Geer [2] and a general expression was determined for all of the δ_j associated with an arbitrary value of N. Examples such as these, where the details can be closely examined, may provide insight as to how general convergence proofs may in the future be constructed. We also leave consideration of rates and types of convergence to future investigations.

The hybrid method in any previous publications known to us has been based on pertur-

bation expansions about a single value of ϵ. In this paper we show several examples where two perturbation expansions, based on widely separated values of ϵ, are used. Work involving expansions about three or more points is in progress and will be reported in future papers. On the basis of the examples studied here, it seems that when expansions about two points are feasible, better results are obtained for values of the parameter between these two points when terms from both expansions are used rather than when terms from only one expansion are used.

The hybrid method may be generalized in the sense that the Bubnov-Galerkin step, wherein the same set of functions is used as "trial" and "test" functions, can be modified as to use different trial and test functions, or even other weighted residual techniques including collocation. Such generalizations are planned for future work. Still other future work will include systems of equations.

The hybrid method based on single perturbation expansion is being applied to a very broad class of problems which includes semi-analytic solutions not only to ordinary differential equations with one parameter, but also to integral equations, to partial differential type equations, to expansions in more than one parameter, to problems discretized by finite differences or finite elements, and to problems with broken symmetry. The research presented here suggests that the hybrid method using multiple perturbation expansions may be useful in these areas as well.
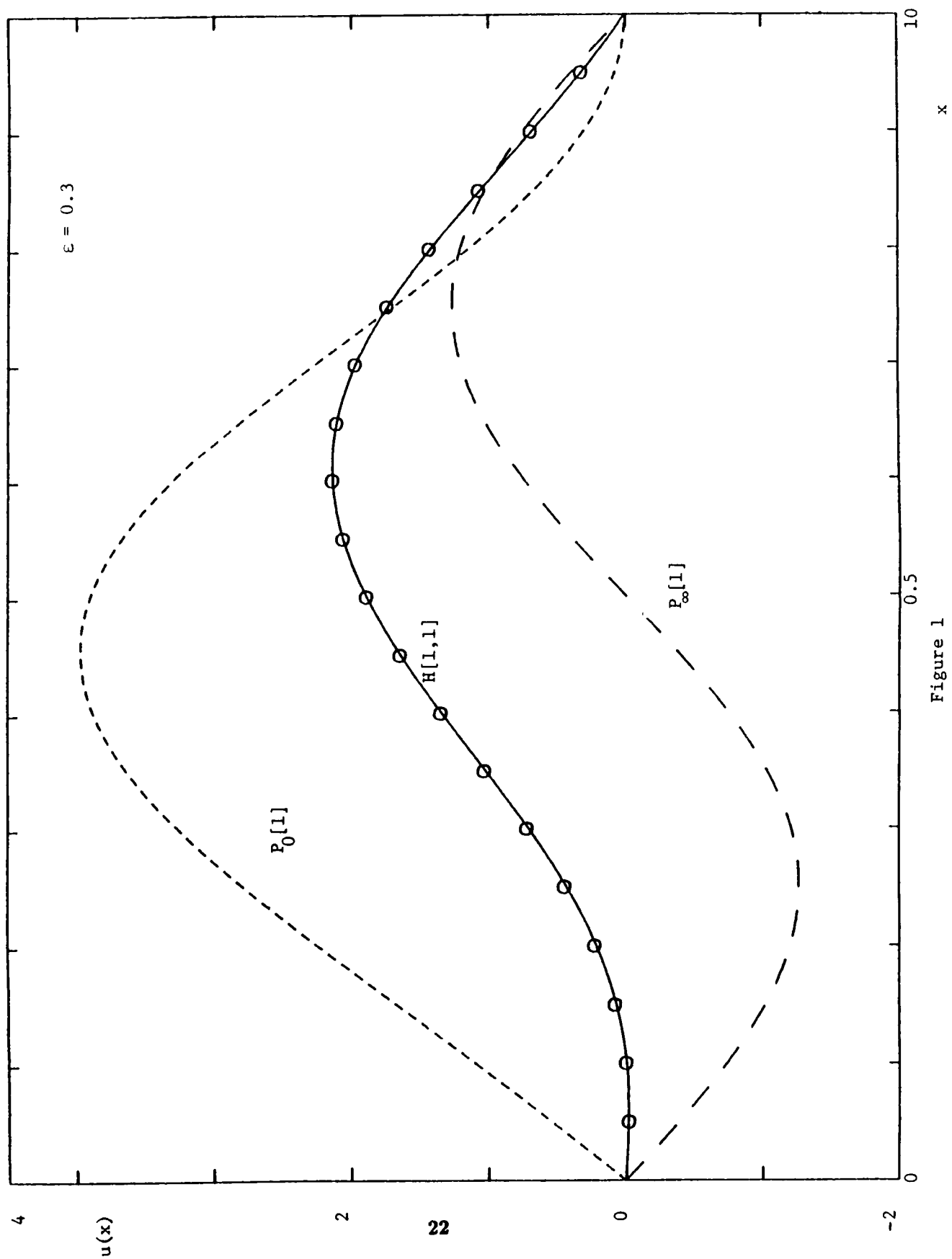
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FIGURE CAPTIONS

1. Perturbation ($P_0[1]$ and $P_\infty[1]$), hybrid ($H[1,1]$) and exact (circles) solutions for the two-point boundary-value problem (3.6) with $\epsilon = 0.3$.

2. Perturbation ($P_0[1]$ and $P_\infty[1]$), hybrid ($H[1,1]$) and exact (circles) solutions for the two-point boundary-value problem (3.7) with $\epsilon = 0.2$.

3. Perturbation ($P_0[1]$, $P_0[2]$, $P_\infty[1]$ and $P_\infty[2]$), hybrid ($H[1,1]$ and $H[2,2]$) and exact (circles) solutions for the two-point boundary-value problem (3.8) with $\epsilon = 0.3$.

4. Perturbation ($P_0[1]$ and $P_\infty[1]$), hybrid ($H[1,1]$) and exact (circles) solutions for the two-point boundary-value problem (4.1) with $a = b = 1$ and $\epsilon = 0.3$.

5. Perturbation ($P_0[1]$ and $P_\infty[1]$), hybrid ($H[1,1]$) and exact (circles) solutions for the wedge slider bearing problem (5.1) with $a = 1$ and $\epsilon = 0.7$.

6. Hybrid ($H[1,1]$) and exact (circles) solutions for the wedge slider bearing problem (5.1) with $a = 1$ and several values of ϵ.

7. The amplitudes δ_1 and δ_2 in the hybrid solution $H[1,1]$ for the wedge slider bearing problem (5.1) with $a = 1$.

8. Perturbation ($P_0[1]$ and $P_\infty[1]$), hybrid ($H[1,1]$) and exact (circles) solutions for the slider bearing problem (5.1) with a parabolic film thickness, $h(x) = 1 + 2(1-x)^2$ and $\epsilon = 0.5$.

9. Regular perturbation ($P_0[1]$ through $P_0[6]$), hybrid ($H[1,0]$ through $H[6,0]$) and exact (circles) solutions for the lowest eigenvalue of the quantum mechanical problem (6.1).

10. Perturbation ($P_\infty[1]$), hybrid ($H[0,1], H[2,0]$ and $H[1,1]$) and exact (circles) solutions for the lowest eigenvalue of the quantum mechanical problem (6.1).

11. Hybrid solutions ($H[1,0]$ through $H[4,0]$ and $H[1,1]$) and exact (circles) solutions the eigenfunction corresponding to the lowest eigenvalue of the quantum mechanical problem (6.1) with $\epsilon = 0.3$.

12. Square of the natural frequency in the nonlinear oscillator problem (7.1) for several values of the parameter μ as computed by the hybrid method (equation (7.6)) and a numerical method.

13. Two-term hybrid ($H[0,2], H[1,1]$ and $H[2,0]$) and exact (circles) solutions to equation (3.6) for $\epsilon = 0.3$.

14. A comparison of relative L_2-error norms as functions of ϵ for several perturbation and hybrid solutions to equation (3.6).
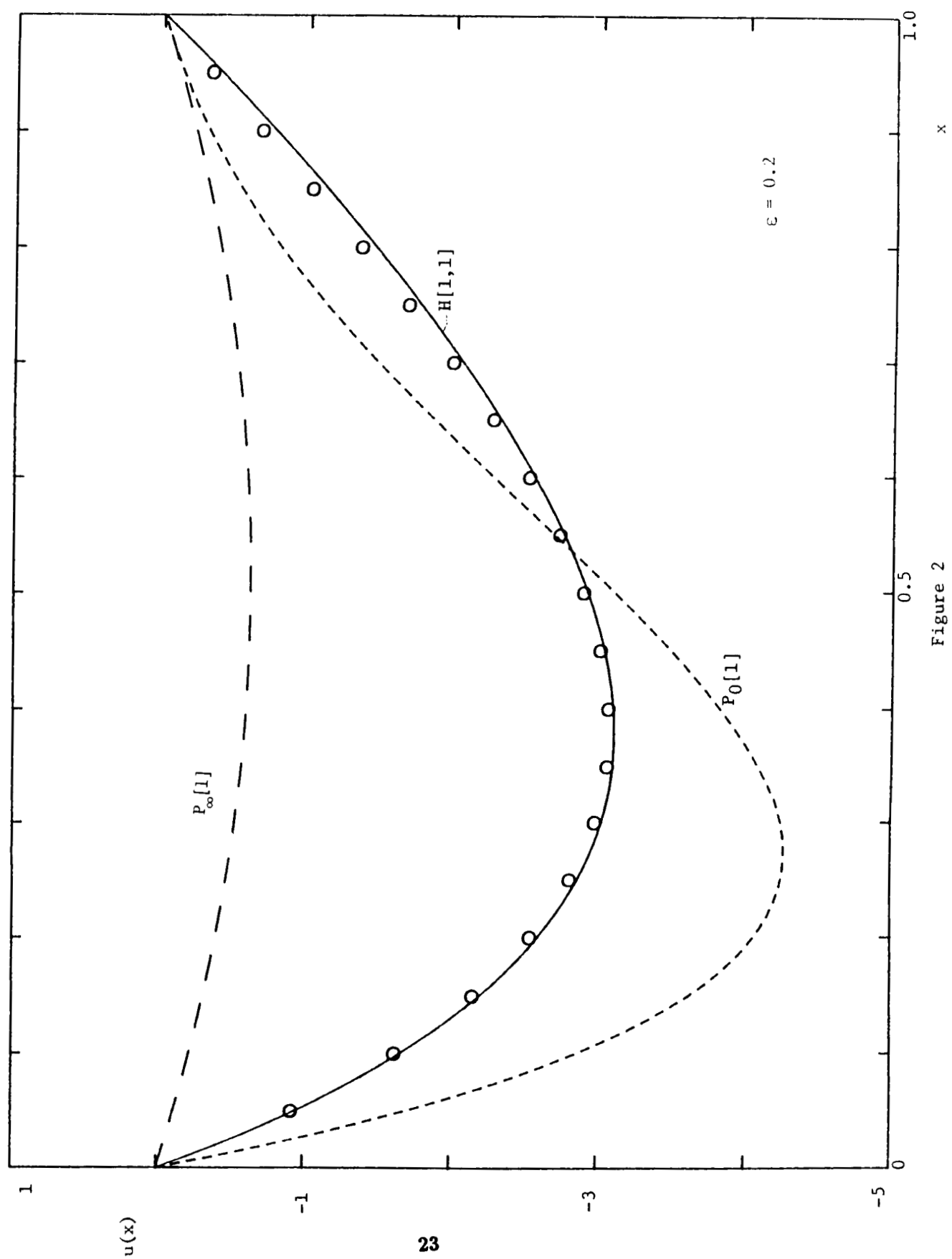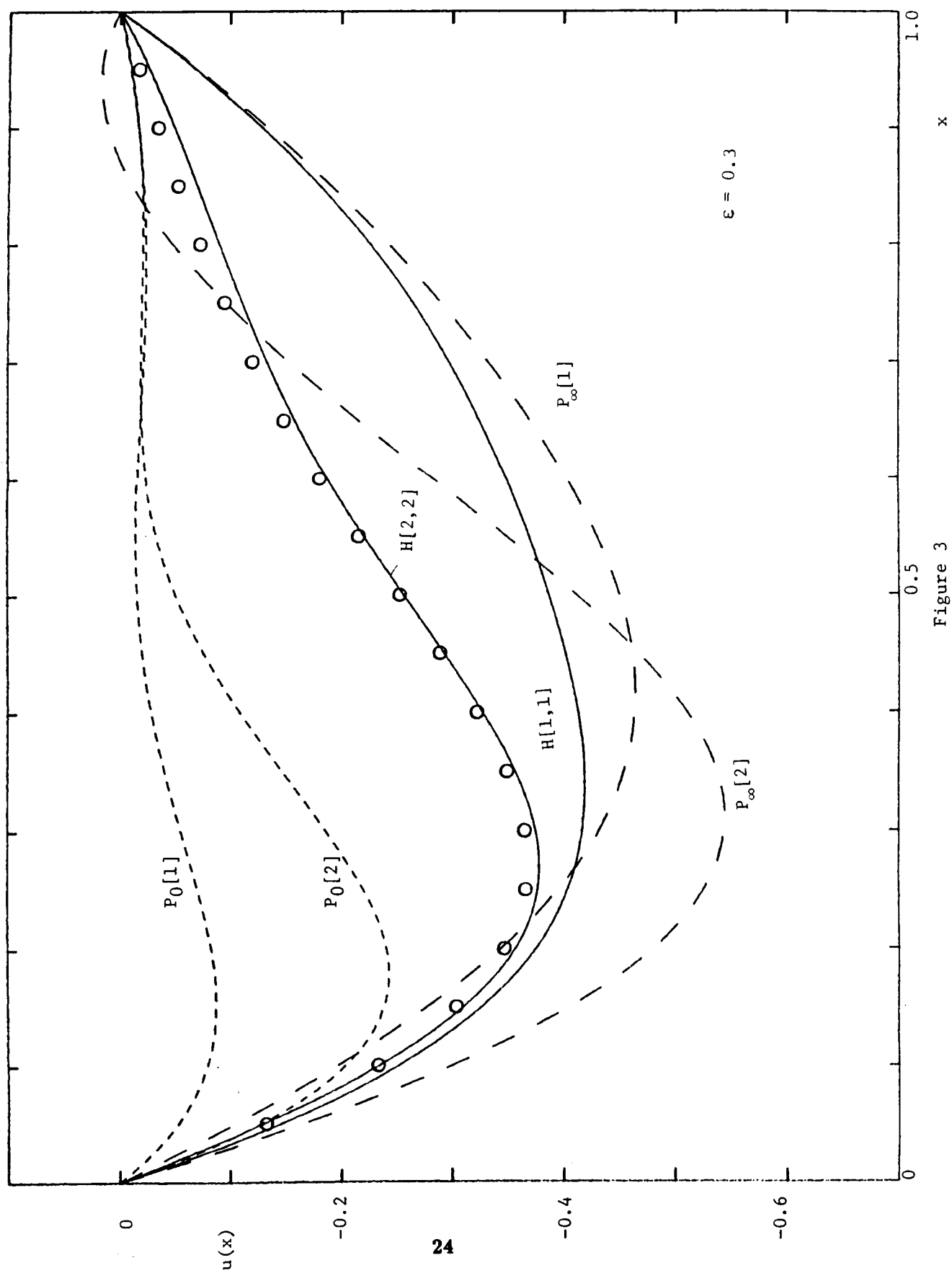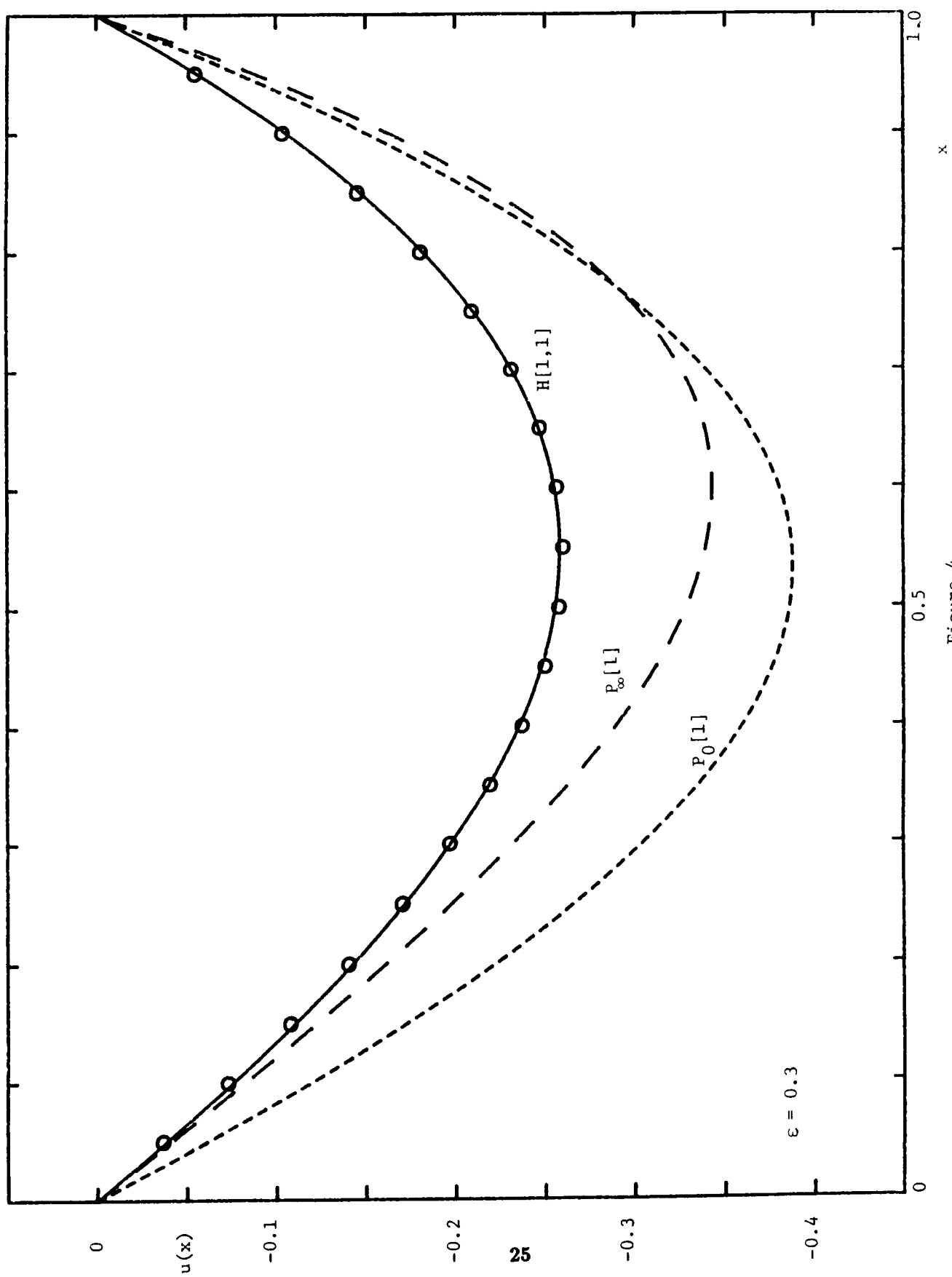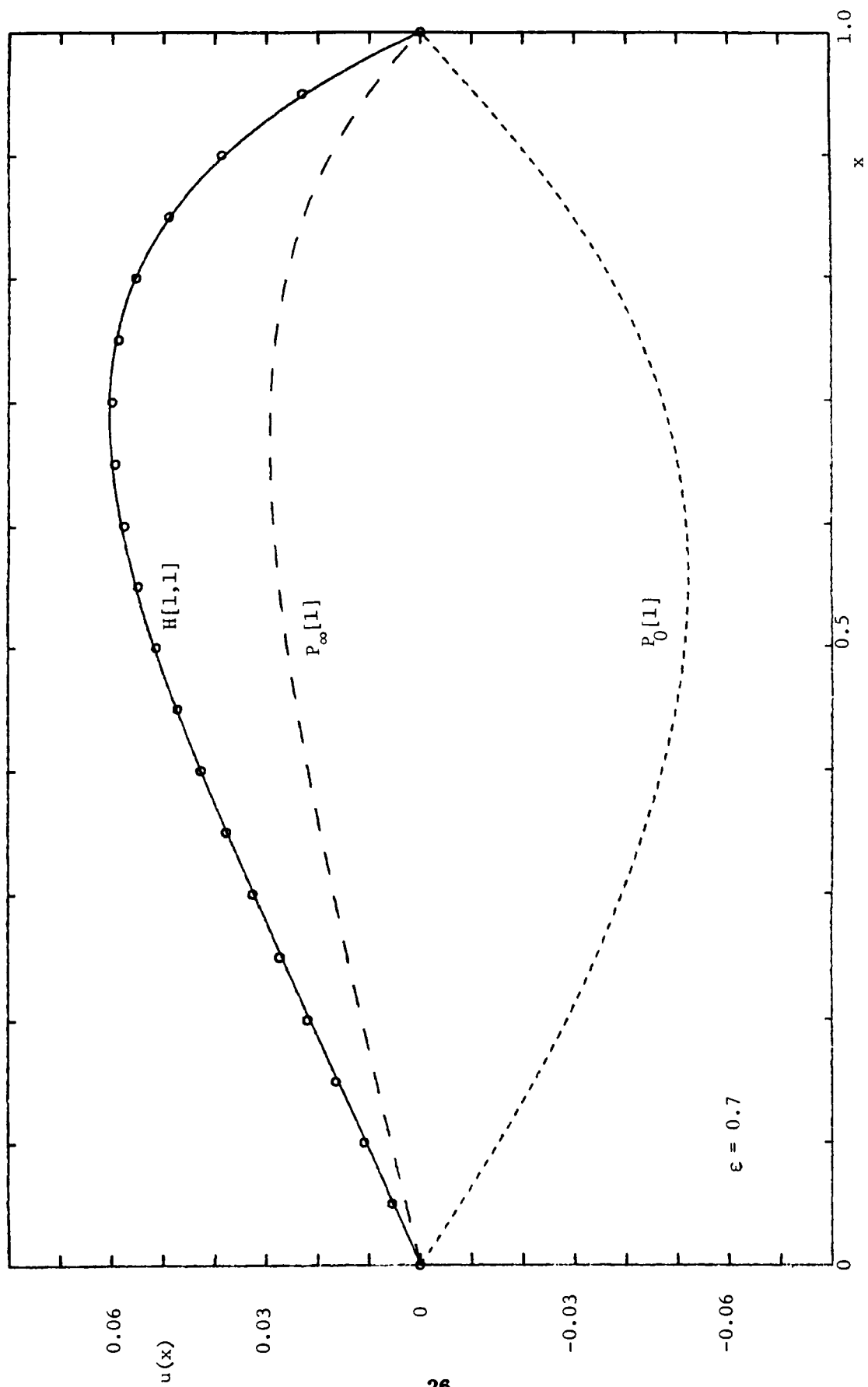
Figure 1

Figure 2

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Figure 3

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Figure 4

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Figure 5

Figure 6

27

Figure 7

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ε = 0.5

$P_0[1]$

$H[1,1]$

$P_\infty[1]$

u(x)

x

0.8

0.6

0.4

0.2

0

0.5

1.0

0

29

Figure 8

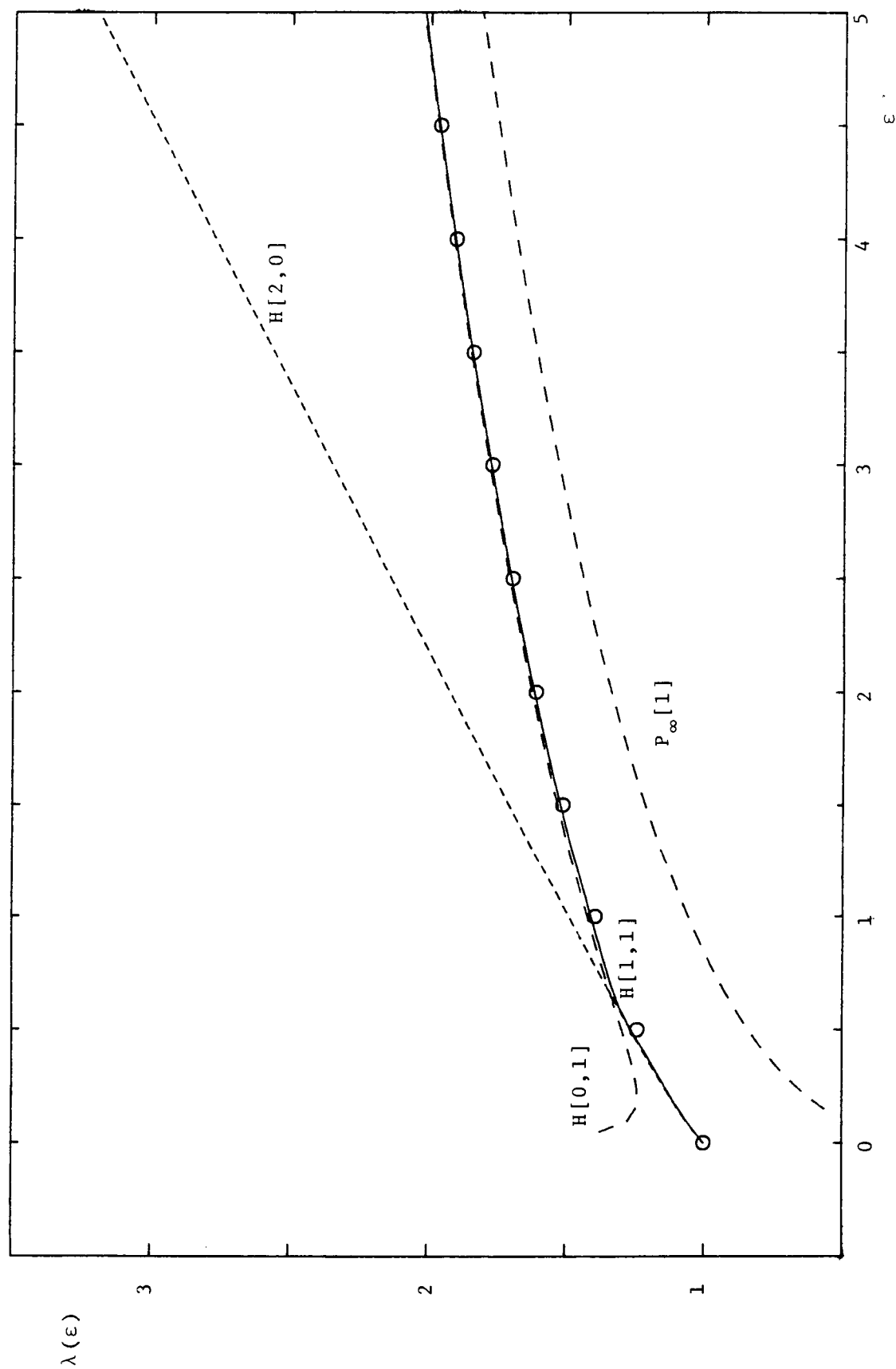Figure 9

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Figure 10

ε = 3.0

u(x)

1.0

0.5

0.0

H[1,1]

H[4,0]

H[3,0]

H[2,0]

H[1,0]

-5.0

0.0
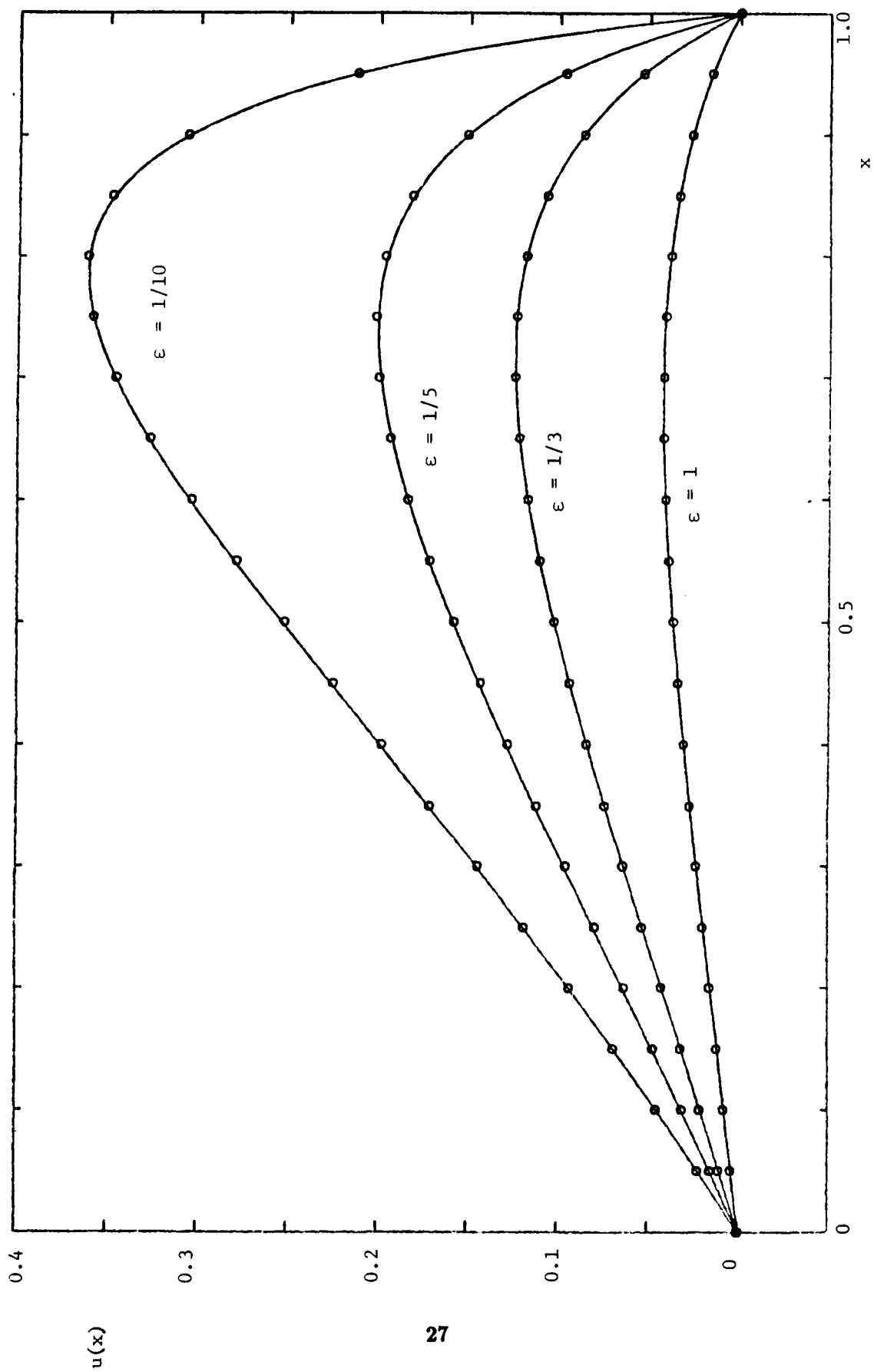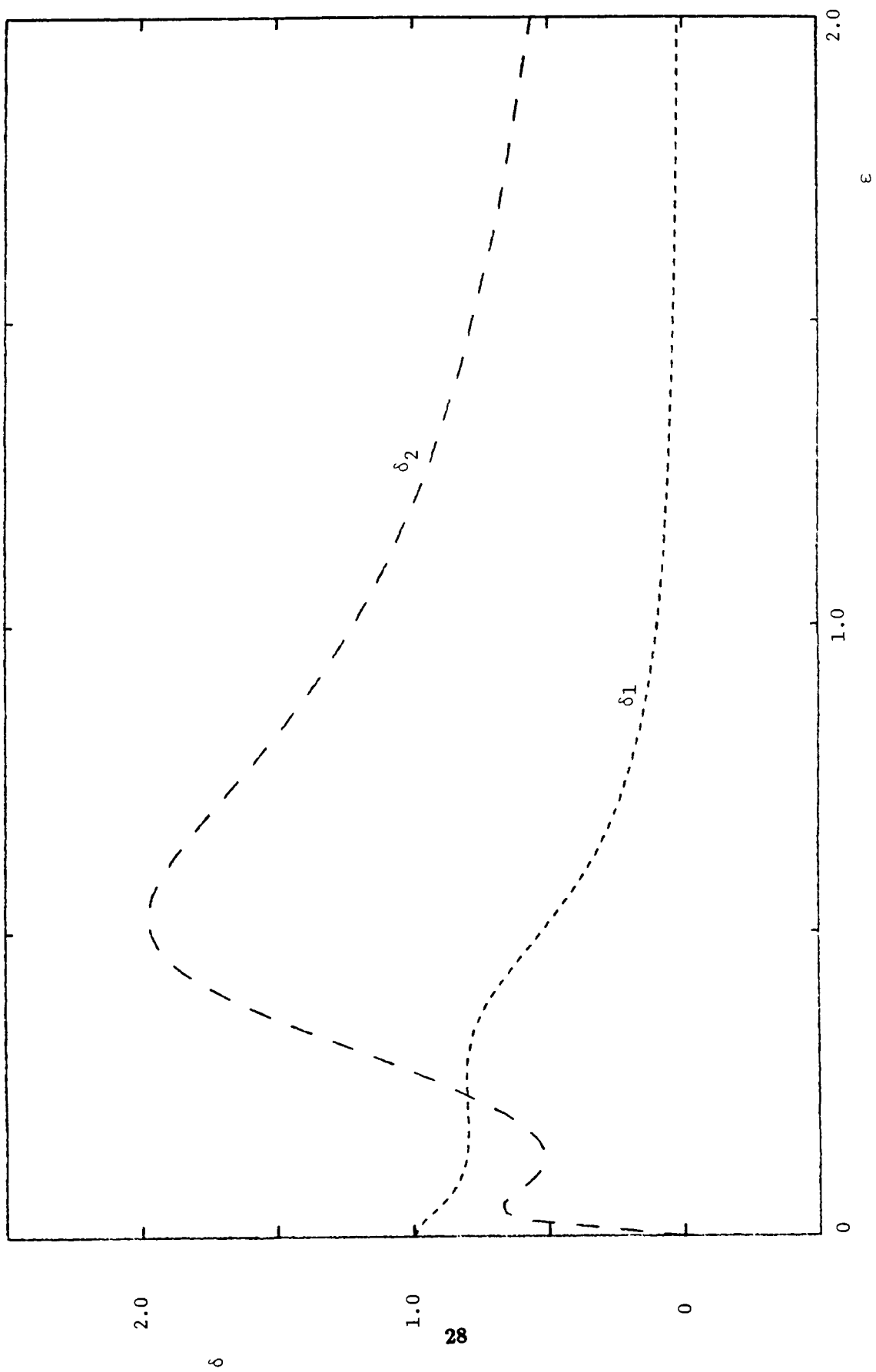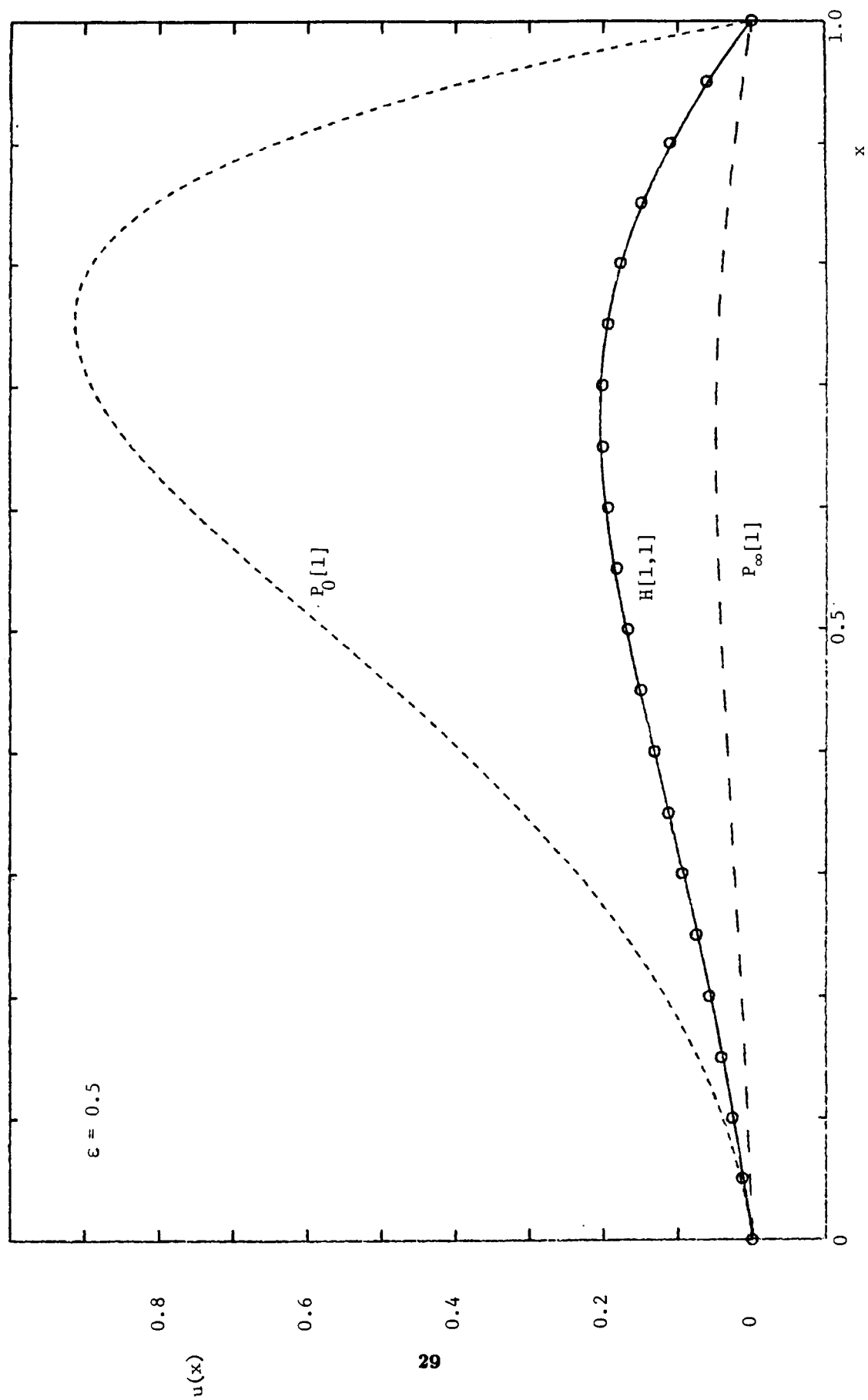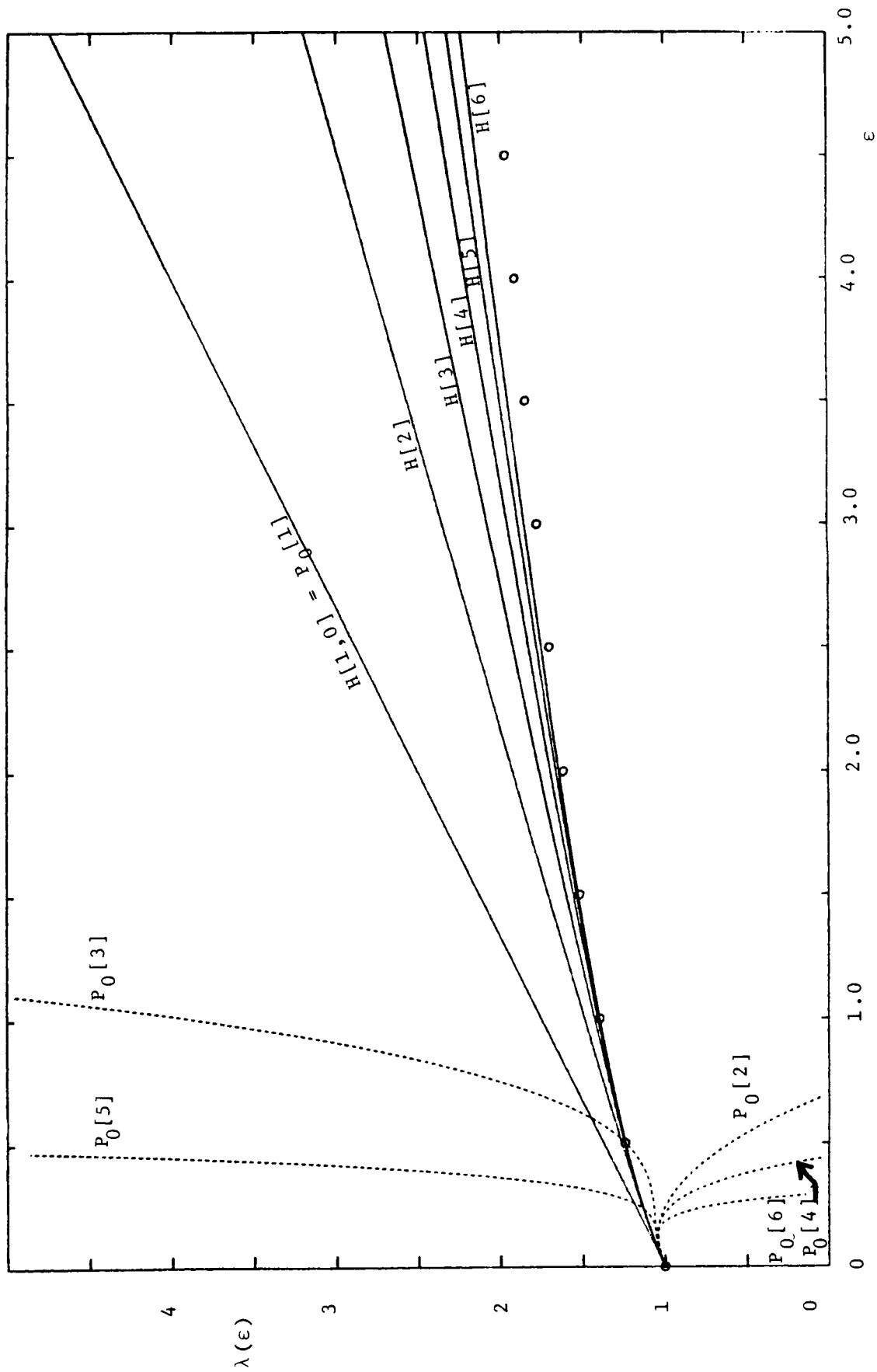
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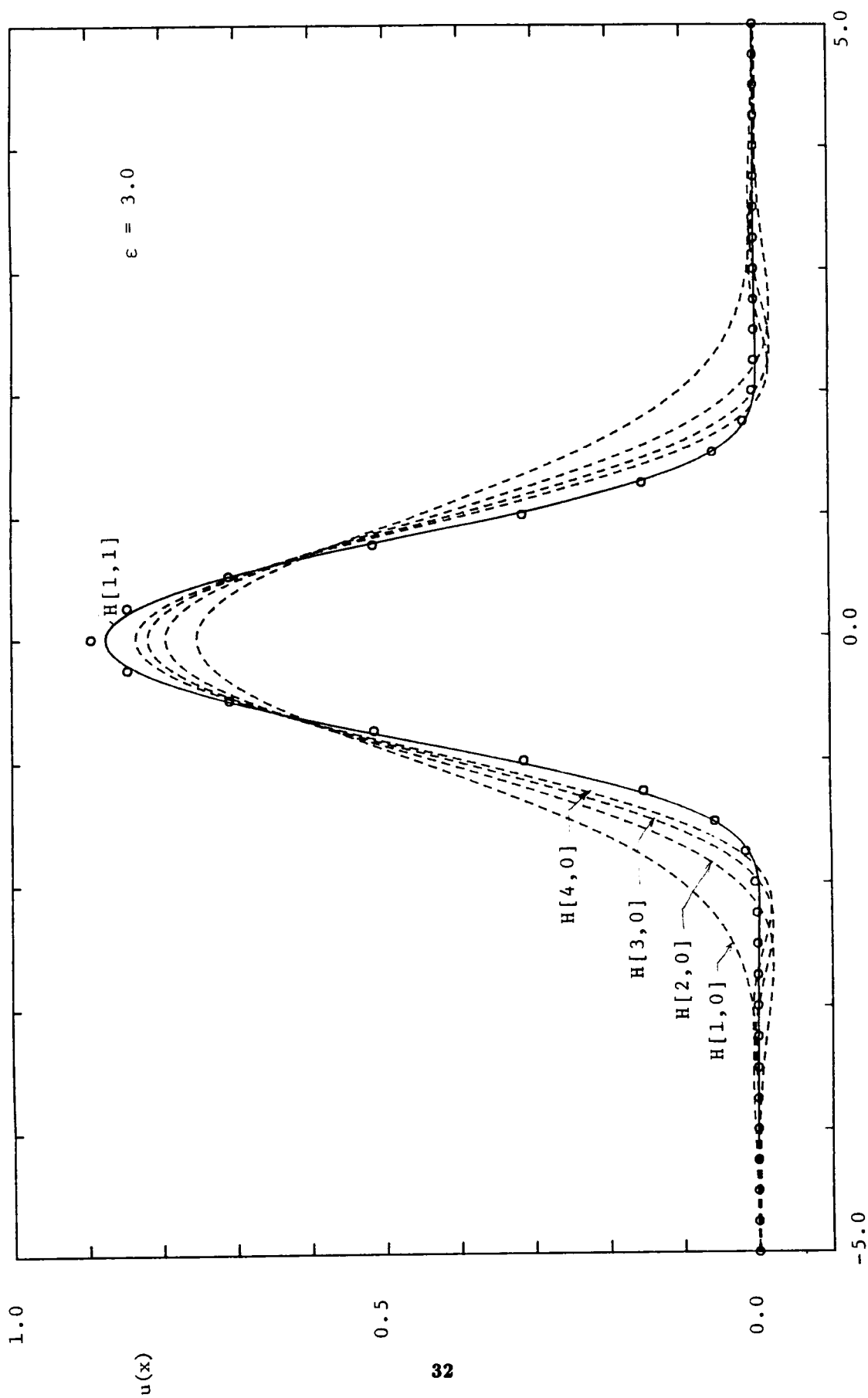
Figure 11

Figure 12

Figure 13

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relative
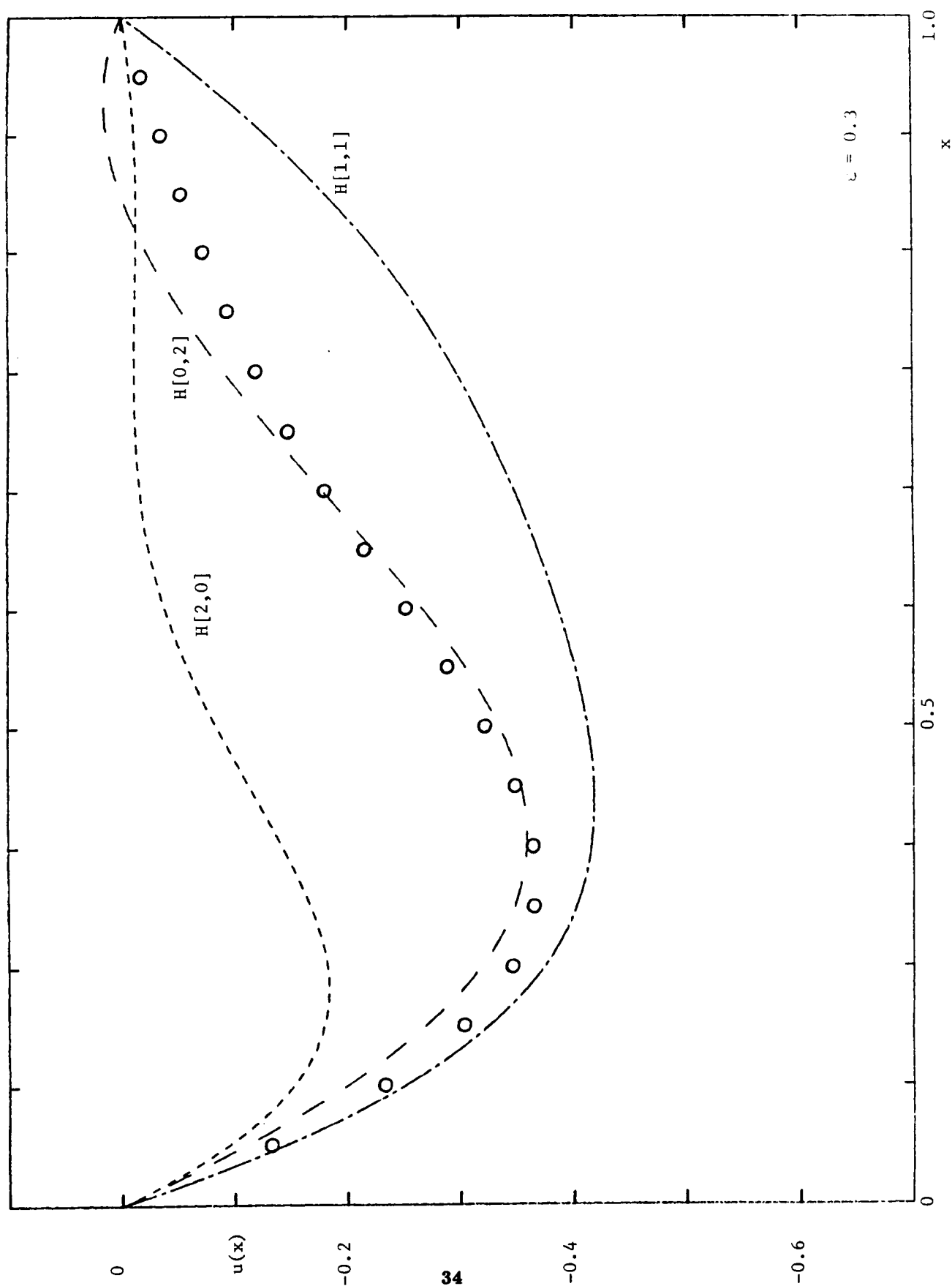L_2-error
norms

Figure 14

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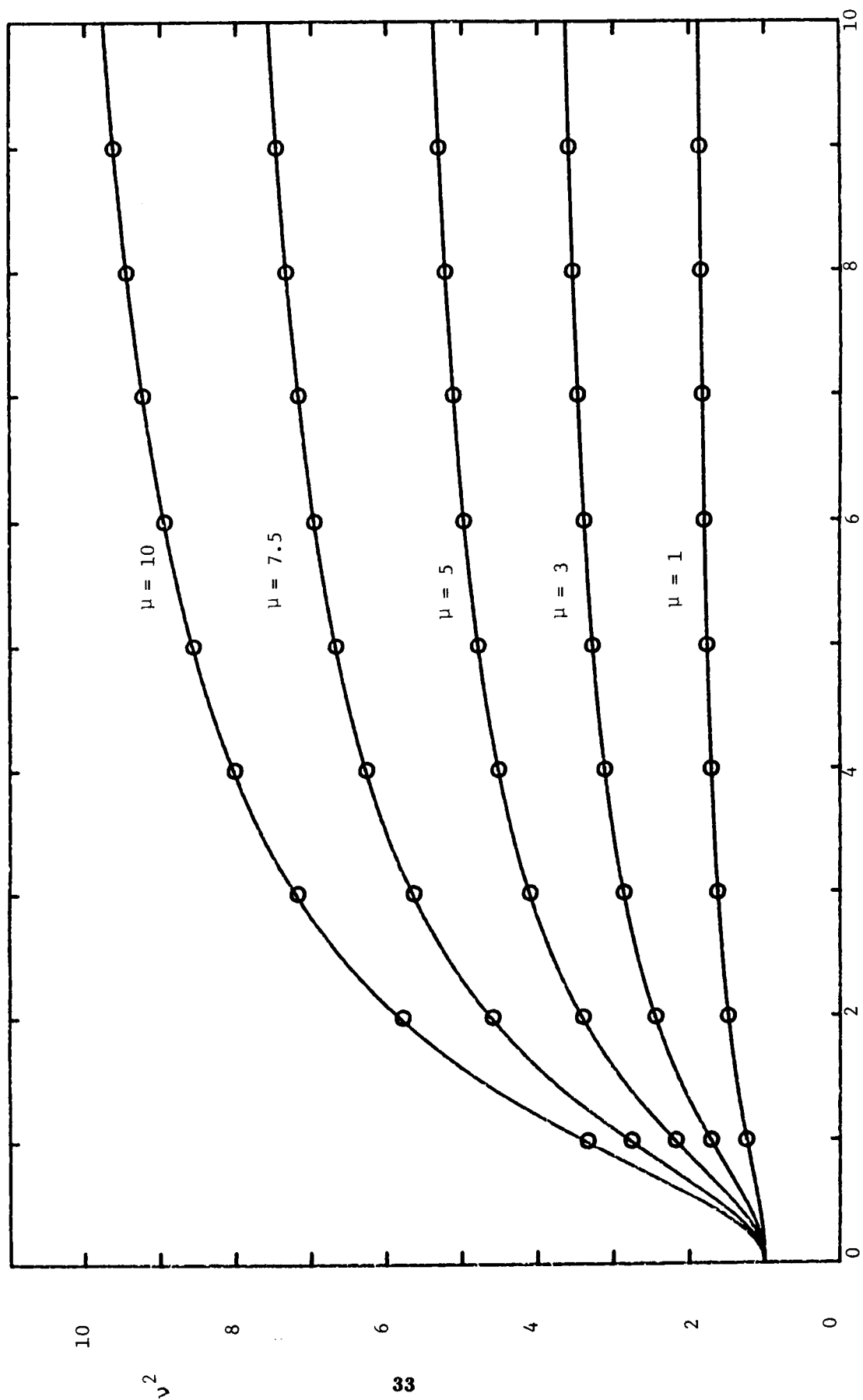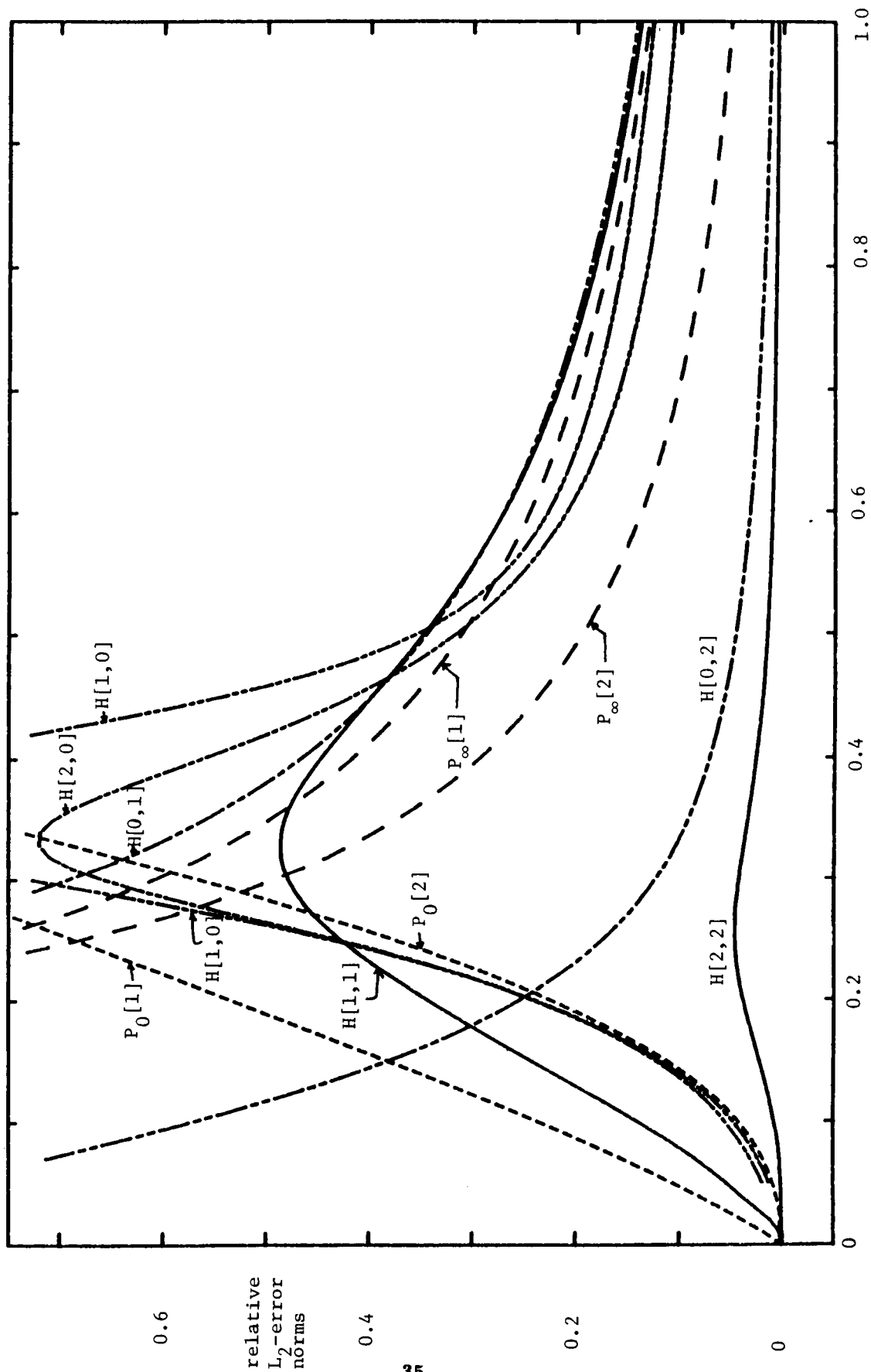
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15. Supplementary Notes

16. Abstract

A two-step hybrid perturbation-Galerkin method for the solution of a variety of differential equations type problems is found to give better results when multiple perturbation expansions are employed. The method assumes that there is a parameter in the problem formulation and that a perturbation method can be sued to construct one or more expansions in this perturbation coefficient functions multiplied by computed amplitudes. In step one, regular and/or singular perturbation methods are used to determine the perturbaton coeffiicient functions. The results of step one are in the form of one or more expansions each expressed as a sum of perturbation coefficient functons multiplied by a priori known gauge functions. In step two the classical Bubnov-Galerkin method uses the perturbation coefficient functions computed in step one to determine a set of amplitudes which replace and improve upon the gauge functions. The hybrid method has the potential of overcoming some of the drawbacks of the pertubation and Galerkin methods as applied separately, while combining some of their better features. In this study the proposed method is applied, with two perturbation expansions in each case, to a variety of model ordinary differential equations problems including: a family of linear two-boundary-value problems, a nonlinear two-point boundary-value prpblem, a quantum mechanical eigenvalue problem and a nonlinear free oscillation problem. The results obtained from the hybrid methods are compared with approximate solutions obtained by other methods, and the applicability of the hybrid method to broader problem areas is discussed.

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